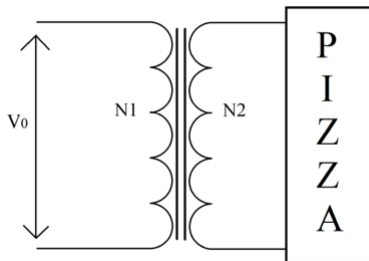

 General Physics: Electromagnetism, Correction 12

Exercise 1 :

A neon sign requires a voltage of $V_2 = 12 \text{ kV}$ to operate. Given an input of $V_0 = 240 \text{ V}$ from a power line, a transformer is used to achieve the necessary voltage. What should be the ratio of the number of turns in the secondary winding to the number of turns in the primary winding of the transformer? What would be the output voltage if the transformer was connected in reverse?

Solution 1 :

An AC transformer consists of two coils of wire wound around a core of iron. The core is used to increase the magnetic flux and to provide a medium for the flux to pass from one coil to the other. For the left part of the transformer we have,

$$V_0 = -N_1 \frac{d\Phi_B}{dt}, \quad (1)$$

where Φ_B is the flux in the core and N_1 the number of turns of the left coil. For the right part of the transformer we have,

$$V_2 = -N_2 \frac{d\Phi_B}{dt}. \quad (2)$$

From the term, $d\Phi_B/dt$, we can connect V_2 to V_0 as,

$$V_0 = N_1 \frac{V_2}{N_2} \rightarrow \frac{V_2}{V_0} = \frac{N_2}{N_1} \rightarrow \frac{N_2}{N_1} = \frac{12 \cdot 10^3}{240} = 50. \quad (3)$$

If the transformer was connected the other way around, we would have a voltage which would be 50 times smaller than the input one:

$$V_2 = \frac{N_1}{N_2} \cdot V_0 = \frac{1}{50} \cdot 240 = 4.8 \text{ V}. \quad (4)$$

Exercise 2 :

A coaxial cable of length l is made of a central conductive wire of radius r_1 and the surrounding tube-like outer conductor of radius r_2 ($r_1 \ll r_2 \ll l$), separated by a dielectric tube (see Figure below). These cables are widely used for transmitting high-frequency signals with the two conductors of the cable on one end connected to a source and on the other end connected to a receiver. Determine the self inductance L of such cable.

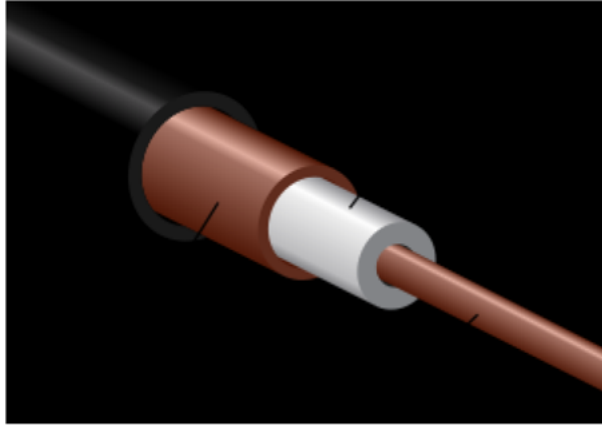


Figure 1: Coaxial cable.

Solution 2 :

This exercise can be solved in two different ways. One is using the fact that the magnetic field is only present between the two conductors and zero outside (because they both carry the same but opposite current) and using the expressions for the **energy stored in the magnetic field** of an inductor. The second way, is to calculate the **magnetic flux** through an area along the cable.

For both ways, we need to know the magnetic field first, which we can find by using Ampère's law:

$$\Phi_B = \oint \mathbf{B} \cdot d\mathbf{s} = \sum I_{\text{encl.}} B = \frac{\mu_0 I_{\Sigma}}{2\pi r}, \quad (5)$$

where I_{Σ} is the total enclosed current and r is the radius of the loop we consider. Here, we are using a constant current I , which is only correct for $r_1 < r < r_2$, but not for $r < r_1$, where the current increases with the loop size as well. For this problem, however, we can ignore this region, because $r_2 \gg r_1$, so the contribution magnetic field within the inner wire is negligible. Since we have a current I flowing in the inner conductor and an equal but opposite current flowing in the shell, $B = \frac{\mu_0 I}{2\pi r}$ between the conductors and $B = 0$ anywhere outside.

1. Using the stored energy

When a current through a conductor (in this case a cable) of inductance L increases from 0 to I , energy is generated and stored in the conductor magnetic field. It relates to the inductance as follows:

$$U = \frac{1}{2}LI^2 \quad (6)$$

One can also express the link between the energy and the magnetic field itself:

$$U = \int \frac{B^2}{2\mu_0} dV, \quad (7)$$

where the expression has to be integrated over the whole volume that contains the field. Since $r_2 \gg r_1$, and we are only interested in the energy stored in the magnetic field, we can neglect the contribution of the magnetic field inside the inner wire, because its volume is much smaller than that between the wire and the shell. Therefore, we end up with:

$$U = \int_{r_1}^{r_2} \frac{B^2(r)}{2\mu_0} dV = \int_{r_1}^{r_2} \frac{B^2(r)}{2\mu_0} 2\pi r l dr = \int_{r_1}^{r_2} \frac{\mu_0^2 I^2}{4\pi^2 r^2 2\mu_0} 2\pi r l dr = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{r_2}{r_1} \quad (8)$$

Then, setting the two initial equations equal:

$$\frac{1}{2}LI^2 = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{r_2}{r_1} \quad (9)$$

Therefore, by isolating the inductance, we get:

$$L = \frac{\mu_0 l}{2\pi} \ln \frac{r_2}{r_1} \quad (10)$$

2. Calculating the flux through a loop formed by the two conductors

In the lecture, we have defined the self inductance L starting from the electromotive force ϵ which we is also related to Faraday's law:

$$\epsilon = -L \frac{dI}{dt} = -\frac{d\phi_B}{dt} \quad (11)$$

In most common cases, the magnetic field is directly proportional to the current. In all these cases we can easily rewrite:

$$L = \phi_B / I. \quad (12)$$

For the second method, we consider a loop formed by connecting the two conductors at their ends. Because $l \gg r_2$, we can neglect the inductance of these additional connections. We end up with a cut along the cable (see brown shaded region in Figure 2), through which a current of I is flowing. At the same time, we have a magnetic flux going through the loop area.

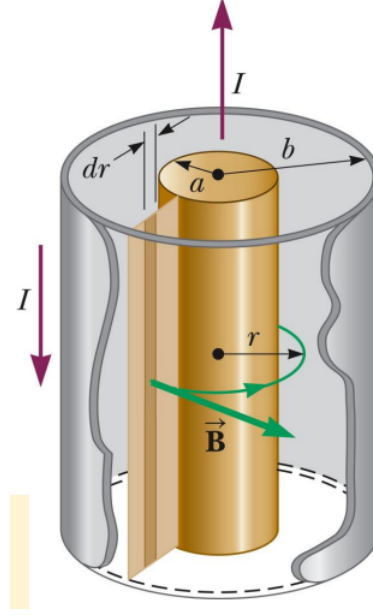


Figure 2: Coaxial cable section

To get the magnetic flux through the loop, we need to integrate the magnetic field over the loop area A , because it is not homogeneous:

$$\Phi_B = \int_A \mathbf{B} \cdot d\mathbf{S} = \int_{r_1}^{r_2} B l \cdot dr = \int_{r_1}^{r_2} \frac{\mu_0 I l}{2\pi r} \cdot dr = \frac{\mu_0 I l}{2\pi} \ln \frac{r_2}{r_1}, \quad (13)$$

where we use $dS = l \cdot dr$ for the infinitesimal area of a short stripe along the whole wire (see dark shaded region in Figure 2) and the expression for B comes from Eq. (5). Also here, we neglect the contribution of the magnetic field in the inner wire (because $r_2 \gg r_1$), by integrating only from r_1 to r_2 . If we would not make this assumption, we would have to find the correct expression for B for that region, as mentioned in the beginning of this solution.

Finally, using Eq. (12), we get the same result as with the first method:

$$L = \frac{\mu_0 l}{2\pi} \ln \frac{r_2}{r_1} \quad (14)$$

Exercise 3 :

Consider a long cable that consists of two parallel straight round wires of length l , diameter $2a$, made of non-magnetic conductive material (see Figure below). The centers of the wires in the cables are separated by distance D , such that: $l \gg D \gg 2a$. When such a cable is used, the directions of the current in the wires are opposite. Derive an expression for self inductance L of the cable.

Hint: To compute the flux, consider a closed loop obtained by connecting with small conductive wires both the extremities of the two different cables. Since these two extra wires are small, you can assume they do not change the self inductance of the two wires. Remember that for calculating the self inductance of a circuit, you have to consider the magnetic flux through the circuit generated by the current flowing in the circuit. Since $D \gg a$, you can neglect the contribution of the magnetic flux through the wires themselves.

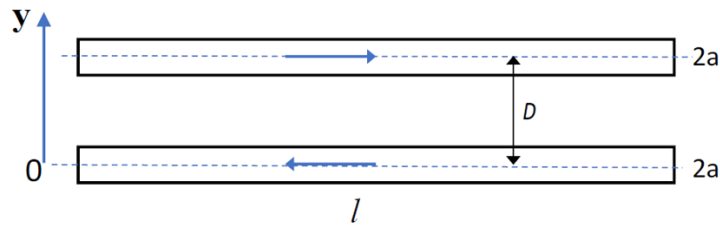


Figure 3: Two wires at distance D .

Solution 3 :

In the lecture, we have defined the self inductance L starting from the electromotive force ϵ which we is also related to Faraday's law:

$$\epsilon = -L \frac{dI}{dt} = -\frac{d\phi_B}{dt} \quad (15)$$

In most common cases, the magnetic field is directly proportional to the current. In all these cases we can easily rewrite:

$$L = \phi_B / I. \quad (16)$$

In this exercise, the wires generate a magnetic field which create circles around the wires with decreasing intensity going further from the wire (distance = r), according to the law:

$$B(r) = \frac{\mu_0 I}{2\pi r} \quad (17)$$

which has been computed by applying the Ampere's Law. In this case we consider only the field outside of the wire, using the fact that $D \gg 2a$.

In order to define the flux of a magnetic field, we need to define a closed loop, or an area over which the magnetic field goes through. We use the hint of the exercises and our closed area will be the one in between the two cables. In this region the magnetic fields generated by the two wires sum up, directed perpendicularly and towards the area (or the sheet).

As the magnetic field changes over the area we are considering, we cannot just say that $\phi_B = BA \cos(\theta)$ but we need to integrate over the surface:

$$\phi_B = \int_S \vec{B} d\vec{S} = \int_a^{D-a} (B_1(y) + B_2(y)) l dy, \quad (18)$$

where $B_1(y) = \frac{\mu_0 I}{2\pi y}$ and $B_2(y) = \frac{\mu_0 I}{2\pi(D-y)}$ is the magnetic field generated by the bottom and top wire respectively. We integrate over dy as the length of the cable l is constant and the magnetic field is changing over the y direction in the picture. Plugging in B_1 and B_2 , we get:

$$\begin{aligned} \phi_B &= \frac{\mu_0 I l}{2\pi} \int_a^{D-a} dy \left(\frac{1}{y} + \frac{1}{D-y} \right) = \frac{\mu_0 I l}{2\pi} [2 \ln(D-a) - 2 \ln(a)] \\ \phi_B &= \frac{\mu_0 I l}{\pi} \ln \left(\frac{D-a}{a} \right) \approx \frac{\mu_0 I l}{\pi} \ln \frac{D}{a}, \end{aligned} \quad (19)$$

where in the last step we have used the fact that $D \gg a$.

Using Eq. (16), we end up with:

$$L = \frac{\mu_0 l}{\pi} \ln \frac{D}{a} \quad (20)$$

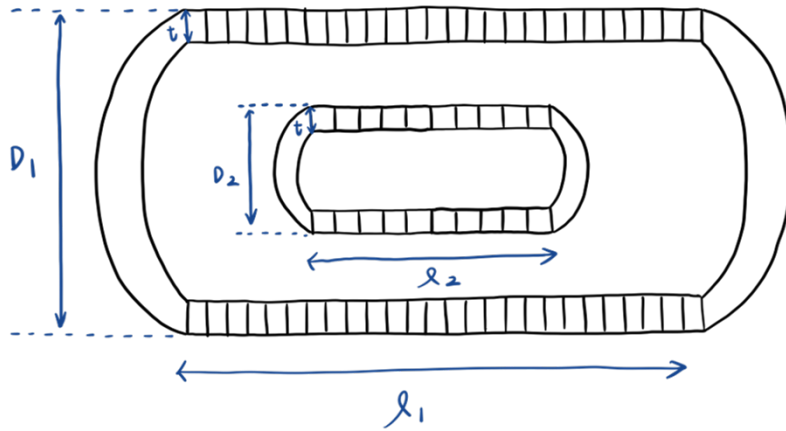


Figure 4: Longitudinal section of solenoids

Exercise 4 :

Two ideal solenoids are made of two pieces of wire of the same thickness $t = 1$ mm but different lengths. Solenoids are tightly wound without gaps between the coils. The larger solenoid of diameter $D_1 = 4$ cm is made from the wire of the length $L_1 = 8$ m, and the smaller solenoid of diameter $D_2 = 2$ cm is made from the wire of the length $L_2 = 2$ m. Note, that this these are not the lengths of the solenoids, but the lengths of the wires used to make them. The smaller solenoid is fully inserted into the larger one. Determine the mutual inductance of the two solenoids M_{21} . $\mu_0 = 4\pi \cdot 10^{-7}$ H/m.

Solution 4 :

Let's label the larger solenoid with 1 and the smaller one with 2. We want to find the mutual inductance $M_{21} = \Phi_{21}/I_1$. The magnetic field in the larger solenoid is given by $B_1 = \mu_0 n_1 I_1$, where $n_1 = N_1/l_1 = \text{\#turns/length}$. The flux inside the smaller solenoid is given by $\Phi_{21} = B_1 A_2 N_2 = (\mu_0 n_1 I_1) A_2 N_2$. The mutual inductance reads

$$\Phi_{21} = M_{21} \cdot I_1 \Leftrightarrow M_{21} = \frac{\Phi_{21}}{I_1} = \mu_0 \cdot n_1 \cdot A_2 \cdot N_2, \quad (21)$$

where

$$A_2 = \pi \cdot \left(\frac{D_2}{2}\right)^2 = \frac{\pi \cdot D_2^2}{4}, \quad N_2 = l_2 \cdot n_2. \quad (22)$$

We finally have

$$M_{21} = \mu_0 \cdot n_1 \cdot \pi \cdot \frac{D_2^2}{4} \cdot l_2 \cdot n_2. \quad (23)$$

Now, we want to find n_1 and n_2 . Since there is no gap between the coils of the wire of thickness t , we have the number of turns over the length:

$$n = \frac{1}{t}, \quad n_1 = n_2 = n \quad (24)$$

Next, we want to compute l_1 and l_2 . We know the total length of the wire. The physical length of the solenoid is the number of turns multiplied by the thickness of the wire, because there is no gap between the coils. We have

$$l_1 = t \cdot \frac{L_1}{\pi \cdot D_1}, \quad l_2 = t \cdot \frac{L_2}{\pi \cdot D_2} \quad (25)$$

where

$$L_2 = \frac{1}{4} \cdot L_1, \quad D_2 = \frac{1}{2} \cdot D_1 \quad (26)$$

It can be convenient (but not strictly necessary) to write l_2 in terms of L_1 and D_1 :

$$l_2 = t \cdot \frac{1}{4} \cdot \frac{L_1 \cdot 2}{\pi \cdot D_1} = \frac{1}{2} \cdot \frac{t}{\pi} \cdot \frac{L_1}{D_1} \quad (27)$$

Substituting in the expression for the mutual inductance we get

$$M_{21} = \mu_0 \cdot n_1 \cdot \pi \cdot \frac{D_2^2}{4} \cdot l_2 \cdot n_2 = \mu_0 \cdot \frac{1}{t} \cdot \pi \cdot \frac{D_1^2}{16} \cdot \left(\frac{1}{2} \cdot \frac{t}{\pi} \cdot \frac{L_1}{D_1} \right) \cdot \frac{1}{t} = \frac{\mu_0 \cdot D_1 \cdot L_1}{32 \cdot t} \quad (28)$$

$$\Leftrightarrow M_{21} = \frac{\mu_0 \cdot (4 \text{ cm}) \cdot (8 \text{ m})}{32 \cdot (1 \text{ mm})} = (10 \text{ m}) \cdot \mu_0 = 1.3 \times 10^{-5} \text{ H}. \quad (29)$$

Exercise 5 :

Figure 5 shows a coil of N_2 turns and radius R_2 surrounding a long solenoid of length l_1 , radius R_1 , and N_1 turns.

1. What is the mutual inductance of the two coils?
2. If $N_1 = 500$ turns, $N_2 = 10$ turns, $R_1 = 3.10$ cm, $l_1 = 75.0$ cm, and the current in the solenoid is changing at a rate of 200 A/s, what is the emf induced in the surrounding coil?

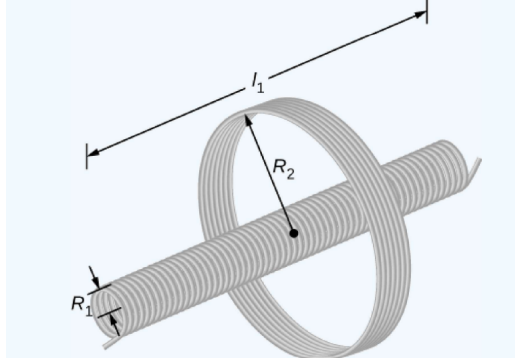


Figure 5: A solenoid surrounded by a coil.

Solution 5 :

There is no magnetic field outside the solenoid, and the field inside has magnitude $B_1 = \mu_0(N_1/l_1)I_1$, directed parallel to the solenoid's axis. We use this magnetic field to determine the magnetic flux through the surrounding coil and then apply this result to calculate the mutual inductance. Finally, the mutual inductance is used to determine the induced emf using the provided expressions.

1. To compute the mutual inductance, we start by computing the magnetic flux Φ_{21} through the surrounding coil, which is given by

$$\Phi_{21} = B_1 \pi R_1^2.$$

Substituting $B_1 = \mu_0(N_1/l_1)I_1$:

$$\Phi_{21} = \frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2.$$

The mutual inductance M is then defined as

$$M = \frac{N_2 \Phi_{21}}{I_1}.$$

Substituting for Φ_{21} gives

$$M = \frac{N_2}{I_1} \left(\frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2 \right) = \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}.$$

Using the given numerical values gives

$$M = \frac{(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}) \times (500) \times (10) \times \pi \times (0.0310 \text{ m})^2}{0.750 \text{ m}} = 2.53 \times 10^{-5} \text{ H}.$$

2. Using Faraday's law, the emf ϵ_2 induced in the surrounding coil is given by:

$$\epsilon_2 = -M \frac{dI_1}{dt}.$$

Substituting the numerical values gives

$$\epsilon_2 = - (2.53 \times 10^{-5} \text{ H}) \times (200 \text{ A/s}) = -5.06 \times 10^{-3} \text{ V}.$$

Thus, the mutual inductance is $M = 2.53 \times 10^{-5} \text{ H}$, and the induced emf in the surrounding coil is $\epsilon_2 = -5.06 \times 10^{-3} \text{ V}$.

Exercise 6 :

Figure 6 shows two long, concentric cylindrical shells of radii R_1 and R_2 . This configuration is a simplified representation of a coaxial cable. The capacitance per unit length of the cable has already been calculated.

1. Determine the magnetic energy stored per unit length of the coaxial cable, and
2. Use this result to find the self-inductance per unit length of the cable.

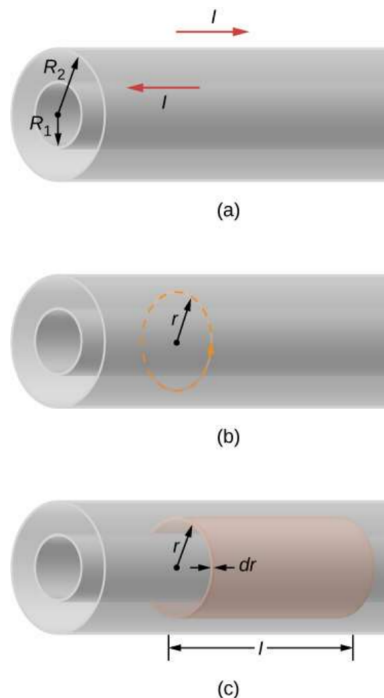


Figure 6: (a) A coaxial cable is represented here by two hollow, concentric cylindrical conductors along which electric current flows in opposite directions. (b) The magnetic field between the conductors can be found by applying Ampère's law to the dashed path. (c) The cylindrical shell is used to find the magnetic energy stored in a length l of the cable.

Solution 6 :

1. We determine the magnetic field between the conductors by applying Ampère's law to the dashed circular path:

$$\oint \vec{B} \cdot d\vec{l} = B(2\pi r) = \mu_0 I. \quad (14.4.6)$$

From this, we find:

$$B = \frac{\mu_0 I}{2\pi r}.$$

In the region outside the cable, Ampère's law shows that $B = 0$, as no net current crosses the area bounded by a circular path where $r > R_2$. Similarly, $B = 0$ for $r < R_1$ (within the inner cylinder). Hence, the magnetic energy is confined between the two conductors.

The energy density of the magnetic field is:

$$u_m = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2}.$$

The energy stored in a cylindrical shell of inner radius r , outer radius $r + dr$, and length l is:

$$dU = u_m(2\pi r l) dr = \frac{\mu_0 I^2}{8\pi^2 r^2} (2\pi r l) dr.$$

The total energy of the magnetic field over the cylindrical shell (for $R_1 \leq r \leq R_2$) is:

$$U = \int_{R_1}^{R_2} dU = \int_{R_1}^{R_2} \frac{\mu_0 I^2}{8\pi^2 r^2} (2\pi r l) dr.$$

Simplifying the integral:

$$U = \frac{\mu_0 I^2 l}{4\pi} \ln \left(\frac{R_2}{R_1} \right).$$

The energy per unit length is:

$$\frac{U}{l} = \frac{\mu_0 I^2}{4\pi} \ln \left(\frac{R_2}{R_1} \right).$$

2. The magnetic energy can be expressed equivalently as

$$U = \frac{1}{2} L I^2, \tag{14.4.7}$$

where L is the self-inductance for a length l of the coaxial cable. Equating the expressions for U , we find:

$$L = \frac{\mu_0 l}{2\pi} \ln \left(\frac{R_2}{R_1} \right).$$

Thus, the self-inductance per unit length is:

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \left(\frac{R_2}{R_1} \right).$$